AD-A080 887 FLORIDA STATE UNIV TALLAHASSEE DEPT OF STATISTICS F/G 12/1
ON THE NORMAL CONVERGENCE OF A CLASS OF SIMPLE BATCH EPIDEMICS.(U)
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UNCLASSIFIED FSU-STATISTICS-M490R ARO-16713.5M NL

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by

10 Naftali A. Langberg

14 FSU-Statistics ~~Report~~-M495R, TR-D-41-ARO
USARO ~~Technical~~ Report No. D-41

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Research supported by the United States Army Research Office, Durham, under Grant No. DAAG29-79-C-0158

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On the Normal Convergence of a Class of Simple Batch Epidemics

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ABSTRACT

A group of n susceptible individuals exposed to a contagious disease is considered. It is assumed that at each instant in time one or more susceptible individuals can contract the disease.

The progress of this epidemic is modeled by a stochastic process $X_n(t)$, t in $[0, \infty)$ representing the number of infective individuals at time t. It is shown that $X_n(t)$, with the suitable standardization and under a mild condition, converges in distribution as $n \to \infty$ to a normal random variable for all t in $(0, t_0)$, where t_0 is an identifiable number.

Key words: Simple batch epidemics, weak convergence, convergence in distribution, normal distributions, Brownian motion, and the Berry-Esseen bound.

1. Introduction and Summary.

We consider a population of susceptible individuals (susceptibles) exposed to contagious disease (disease). We say that the population of susceptibles undergoes a simple epidemic if the following three assumptions hold. [Bailey (1975)].

(1.1) Once a susceptible contracts the disease he remains contagious during the duration of the epidemic.

(1.2) Individuals neither join nor do they depart from the population, and

(1.3) At each point in time at most one susceptible contracts the disease.

It is quite conceivable that when an infective individual (infective) interacts with a group of susceptibles one or more of them contract the disease. In the paper we consider such a situation. We say that a population of susceptibles, exposed to a disease, undergoes a simple batch epidemic if Assumptions (1.1) and (1.2) hold and if the following holds:

(1.4) At each point in time one or more susceptibles can contract the disease.

Let denote by $\underline{T_0}$ the time the first group of susceptibles contracts the disease, and let $\underline{n}$ be the number of susceptibles at T_0. We describe the progress of the simple batch epidemic among the susceptibles by a stochastic process $X_n(t)$, t in $[0, \infty)$ representing the number of infectives at time t measured from T_0. In Section 2 we construct a variety of stochastic processes that model the progress of simple batch epidemics. However, in the sequel we restrict our analysis to a special class of simple batch epidemics. The stochastic processes corresponding to this class of simple batch epidemics are presented in the last paragraph of Section 2. This class of simple batch epidemics generalizes models used and motivated by Severo (1969) to describe simple epidemic situations.

Billard, Lacayo and Langberg (1980) consider a different class of simple batch epidemics. They prove, for this class, that the number of infectives at time t: $X_n(t)$, t in $[0, \infty)$ converges in distribution as $n \to \infty$ to an identifiable discrete random variable (rv) for all t in $[0, \infty)$. In Section 4 we consider the class of simlple batch epidemics defined in the last paragraph of Section 2. We assume that

(1.5) $\lim_{n\to\infty}(n^{-1}X_n(0) - \lambda)\sqrt{n} = \Delta$, where λ is in $[0, \infty)$ and Δ is in $(-\infty, \infty)$,

and prove that $X_n(t)$, with the suitable standartization, converges in distribution as $n \to \infty$ to a normal rv for all t in $(0, t_0)$ and identify t_0.

In Section 3 we present some key lemmas used later in the proof of our main result given in Section 4. Throughout we define a sum over an empty set of indiciesas zero, and denote by L(n), n = 1, 2, ..., a sequence of integers assuming for almost all n values in the set {1, ..., n} respectively.

2. Model Construction

In this section we construct stochastic processes that describe the progress of simple batch epidemics among the susceptibles. We need some notation.

Let $Z_1, Z_1, \ldots,$ be a sequence of i.i.d rv's assuming values in the set $\{1, 2, \ldots\}$, let $EZ_1 = m$, and let $\text{Var } Z_1 = \sigma^2$. Let $D_k = \Sigma_{q=1}^{k} Z_q$, and let $\tau(k) = \min\{q: D_q \geq k, q = 0, \ldots, k\}$ for all k in $\{0, 1, \ldots\}$. Further, let $U_1, U_2, \ldots,$ be an i.i.d sequence of nonnegative rv's independent of the sequence $Z_1, Z_2, \ldots,$ let $EU_1 = 1$, let $\text{Var } U_1 = \delta^2$, and let us assume that $EU_1^3 < \infty$. Finally, let $T_{n,k}$, $k = 1, \ldots, \tau(n)$ be the k^{th} interinfection time defined as the time that elapses between the $(k-1)^{th}$ and the k^{th} change in the number of infectives, let $S_{n,k} = \Sigma_{q=1}^{\tau(k)} T_{n,q}$, $k = 0, \ldots, n$, let $S_{n,n+1} = \infty$, and let $\mu(n, q)$, $q = X_n(0), \ldots, X_n(0) + n - 1$, be positive real numbers.

We are ready now to construct the desired stochastic processes. Let $k = 0, \ldots, n$ and let t be in $[0, \infty)$. Then the following event equality holds.

$$(X_n(t) - X_n(0) = k) = (S_{n,k} \leq t < S_{n,k+1}). \tag{2.1}$$

Thus, for all $t_1, \ldots, t_e$ in $(0, \infty)$ and all e in $\{1, 2, \ldots\}$ the distribution function of the random vector $\{X_n(t_1), \ldots, X_n(t_e)\}$ is determined by Equation (2.1). Consequently, to construct the process $X_n(t)$, t in $[0, \infty)$ it suffices to determine the distribution function of the random vector $\{\tau(n), T_{n,1}, \ldots, T_{n,\tau(n)}\}$. To determine the distribution function of this random vector it is enough to present the distribution function of the conditional random

vector $\{T_{n,1}, T_{n,\tau(n)}\}|\tau(n)$. We assume throughout that

(2.2) the two conditional random vectors $\{T_{n,1}, \ldots, T_{n,\tau(n)}\}|\tau(n)$ and $\{\mu(n, X_n(0))U_1, \mu(n, X_n(0) + D_1)U_2, \ldots, \mu(n, X_n(0) + D_{\tau(n)-1})U_{\tau(n)}\}|\tau(n)$ are equal in distribution for $n = 1, 2, \ldots$.

Let U_1 be an exponential random variable. Then the differential equations associated with the corresponding simple batch epidemics have a relative simple form. Although these differential equations are not used in the paper we present them for the sake of completeness.

Proposition 2.1. Let $P_{n,q}(t) = P\{X_n(t) - X_n(0) = q\}$, $q = 0, \ldots, n$, t in $[0, \infty)$. Let us assume that U_1 is an exponential rv. Then for all t in $[0, \infty)$

$$(2.3)\quad \frac{dP_{n,k}(t)}{dt} = \begin{cases} -\mu^{-1}(n,b_n)P_{n,o}(t) & k=0 \\ -\mu^{-1}(n,b_n+k)P_{n,k}(t)+\sum\limits_{q=o}^{k-1}\mu^{-1}(n,b_n+q)P_{n,q}(t)P\{Z_1=k-q\} & 0<k<n \\ \sum\limits_{q=o}^{n-1}\mu^{-1}(n,b_n+q)P_{n,q}(t)P\{Z_1\geq n-q\} & k=n. \end{cases}$$

Finally, we present the class of stochastic process corresponding to the simple batch epidemics that are the subject of our analysis. Let $\mu(n, q)$, $q = X_n(0), \ldots, X_n(0) + n - 1$, be positive real numbers given by:

$$(2.4)\qquad \mu(n, q) = Aq^{-\beta}(n + X_n(0) - q)^{-\alpha} n^{\alpha+\beta-1}.$$

We assume that A, α are in $(0,\infty)$ and that the range of β depends on the value of λ in Condition (1.5) as follows: for $\lambda > 0$ β is in $(0,\infty)$ and for $\lambda = 0$ β is in $(0, 1/3)$. In the sequel we adress ourselves to simple batch epidemics defined by Equations (2.1), (2.2) and (2.4).

3. Preliminaries.

Let $f(z, a) = \int_0^{m^{-1}z} (1-mp)^{-\alpha a}(\lambda + mp)^{-\beta a}dp$ for z in [0, 1) and for a in [0, ∞), let $J(L(n), a) = n^{a(\alpha+\beta-1)}\sum_{q=1}^{\tau(L(n))}(n - D_{q-1})^{-\alpha a}(X_n(0) + D_{q-1})^{-\beta a}$, $n = 1, 2, \ldots$, a in [0, ∞), and let $[y]$ be the largest integer less than or equal to y, y in (−∞, ∞). Further, let $W_n(p) = (D_{[np]} - mpn)n^{-\frac{1}{2}}$, $n = 1, 2, \ldots$, p in [0, 1], let denote by W(p), p in [0, 1] a normalized Brownian motion [Breiman (1968), p. 257], and let I denote the indicator function. Throughout we assume that

(3.1) $\lim_{n\to\infty}(n^{-1}L(n) - z)\sqrt{n} = v$, where z is in (0, 1) and v is in (−∞, ∞).

In this section we present three key lemmas. We need these lemmas in the next section in order to obtain our main result. First, we show that the process $W_{L(n),1}(p) = (n^{1}\tau(L(n)) - zm^{-1})\sqrt{n} + \sigma W_n(p)$, $n = 1, 2, \ldots$, p in [0, 1] converges weakly as $n \to \infty$ to the process $W_1(p) = m^{-1}\{v - \sigma W(m^{-1}z)\} + \sigma W(p)$, p in [0, 1]. Next, we show that $n^{a-1}J(L(n),a)$ converges in probability as $n \to \infty$ to f(z, a). Finally, we show that $\{J(L(n), 1) - f(z, 1)\}\sqrt{n}$ converges in distribution as $n \to \infty$ to $m^{-1}\{v - \sigma W(zm^{-1})\}(1 - z)^{-\alpha}(\lambda + z)^{-\beta}$ $- \beta\Delta I(\lambda > 0)f(z, 1) + \sigma\int_0^{zm^{-1}} W(p)(1 - mp)^{-\alpha}(\lambda + mp)^{-\beta}\{\alpha(1 - mp)^{-1} - \beta(\lambda + mp)^{-1}\}dp.$

Without loss of generality we can assume that

(3.2) the process W(p) has continuous sample paths [Breiman (1968), p. 259],

and that

(3.3) the rv $\sup_{0\le p\le 1}|W_n(p) - \sigma W(p)|$ converges in probability as $n \to \infty$ to zero.

[Brieman (1968), p. 280]

We are ready now to establish the weak convergence of the process $W_{L(n),1}(p)$.

Lemma 3.1. Let us assume that Condition (3.1) holds. Then $W_{L(n),1}(p)$ converges weakly as $n \to \infty$ to $W_1(p)$.

Proof. By Statement (3.2) the process $W_1(p)$ has continuous sample paths. Further for all $0 \le p_1 \le p \le p_2 \le 1$

$$E(W_{L(n),1}(p) - W_{L(n),1}(p_1))^2(W_{L(n),1}(p_2) - W_{L(n),1}(p))^2 =$$

$$= E(W_n(p) - W_n(p_1))^2(W_n(p_2) - W_n(p))^2 \le 4\sigma^2(p_2 - p_1)^2$$

[Billingsley (1968), p. 138, (16.4)] Thus, to prove the result of the lemma it suffices to show that:

(3.4) the random vector $\{W_{L(n),1}(p_1), \ldots, W_{L(n),1}(p_e)\}$ converges in distribution as $n \to \infty$ to the random vector $\{W_1(p_1), \ldots, W_1(p_e)\}$ for all $p_1, \ldots, p_e$ in $[0, 1]$ and for all e in $\{1, 2, \ldots\}$ [Billingsley (1968), p. 128]

Next, to prove Statement (3.4) it is enough to show that:

(3.5) the random vector $\{(n^{-1}\tau(L(n)) - m^{-1}z)\sqrt{n}, W_n(p_1), \ldots, W_n(p_e)\}$ converges in distribution as $n \to \infty$ to the random vector $\{m^{-1}(v - \sigma W(zm^{-1})), \sigma W(p_1), \ldots, \sigma W(p_e)\}$ for all $p_1, \ldots, p_e$ in $[0, 1]$ and all e in $\{1, 2, \ldots\}$ by the Carmer-Wald device [Billingsley (1968), p. 49].

Finally, we prove Statement (3.5). Let $x, x_1, \ldots, x_e$ be in $(-\infty, \infty)$, let $p_1, \ldots, p_e$ be in $[0, 1]$, and let $H = [xn^{\frac{1}{2}} + zm^{-1}n]$. We note that

$$P\{(n^{-1}\tau(L(n)) - m^{-1}z)\sqrt{n} \le x, W_n(p_r) \le x_r, r = 1, \ldots, e\} =$$

$= P\{D_H \ge L(n), W_n(p_r) \le x_r, r = 1, \ldots, e\}$, and that by Condition

(3.1) $\lim_{n\to\infty}(L(n) - mH)n^{-\frac{1}{2}} = v - mx$. Since,

$\lim_{n\to\infty} \mathrm{Var}\{n^{-\frac{1}{2}}(D_H - mH - D_{[nzm^{-1}]} + nz)\} = 0$ we conclude that

$$\lim_{n\to\infty} P\{(n^{-1}\tau(L(n)) - m^{-1}z)\sqrt{n} \le x, W_n(p_r) \le x_r, r = 1, \ldots, e\} =$$

$$\lim_{n\to\infty} P\{W_n(zm^{-1}) \ge v - mx, W_n(p_r) \le x_r, r = 1, \ldots, e\}.$$

Consequently Statement (3.5) follows by Statement (3.3) and a well known result [Billingsley (1968), p.25, Th. 4.1.]. ||

In particular we obtain from Lemma 3.1

Corollary 3.2. Let us assume that Condition (3.1) holds. Then (a) $(n^{-1}\tau(L(n)) - m^{-1}z)\sqrt{n}$ converges in distribution as $n \to \infty$ to $m^{-1}\{v - \sigma W(zm^{-1})\}$ and (b) $n^{-1}\tau(L(n))$ converges in probability as $n \to \infty$ to $m^{-1}z$.

For the sake of completeness we note that $n^{-1}\tau(L(n))$ converges with probability 1 as $n \to \infty$ to $m^{-1}z$ provided $\lim_{n\to\infty} n^{-1}L(n) = z$ in $(0, 1]$.

Now, we establish the convergence of $n^{a-1}J(L(n), a)$ to $f(z, a)$. We need some notation and one simple result. Let $I_n(p) = I(p < n^{-1}\tau(L(n)))$, and let $V_n(p) = n^{-\frac{1}{2}}W_n(p)$, $n = 1, 2, \ldots$, p in $[0, 1]$. Since, $J(L(n), a) =$ $n^{a(\alpha+\beta-1)}\int_0^{\tau(L(n))} (n - D_{[x]})^{-\alpha a}(X_n(0) + D_{[x]})^{-\beta a}dx$ we obtain by the substitution $x = np$ that

$$(3.6) \qquad n^{a-1}J(L(n), a) =$$

$$= \int_0^{n^{-1}\tau(L(n))} (1 - mp - V_n(p))^{-\alpha a}(n^{-1}X_n(0) + mp + V_n(p))^{-\beta a}dp.$$

We are ready now to establish the convergence of $n^{a-1}J(L(n), a)$.

Lemma 3.3. Let us assume that Conditions (1.5) and (3.1) hold. Then $n^{a-1}J(L(n), a)$ converges in probability as $n \to \infty$ to $f(z, a)$ for all a in $[0, 3]$.

Proof. First we note that for all p in $(0, n^{-1}\tau(L(n)))$ and almost all n in $\{1, 2, \ldots\}$

$$(3.7) \qquad 2 \geq 2 - p \geq 1 - mp - V_n(p) \geq (1 - z)/2, \text{ and}$$

$$(3.8) \qquad 2(\lambda + z) \geq n^{-1}X_n(0) + mp + V_n(p) \geq \lambda/2 + p.$$

By Statement (3.3) $V_n(p)$ converges in probability as $n \to \infty$ to zero for all p in $[0, 1]$. By Corollary 3.2 (b) $I_n(p)$ converges with probability 1 as $n \to \infty$ to $I(p < m^{-1}z)$ on $[0, 1] - \{m^{-1}z\}$. Further, for almost all n $J(L(n), a) = \int_0^1 I_n(p)(1 - mp - V_n(p))^{-\alpha}(n^{-1}X_n(0) + p + V_n(p))^{-\beta}dp$. Consequently the result of the lemmas follows by the dominated convergence theorem [**Loève** (1963), p. 125]. ||

We note that Lemma 3.3 remains valid if Condition (3.1) is replaced by the weaker condition that $\lim_{n\to\infty} n^{-1}L(n) = z$ in (0, 1).

From Inequalities (3.7) and (3.8) we conclude that for almost all n in $\{1, 2, \ldots\}$

(3.9) $$n^{a-1}J(L(n), a) \leq (1 - z)^{-\alpha a}2^{\alpha a}\int_0^{n^{-1}\tau(L(n))}(\lambda/2 + p)^{-\beta a}dp,$$ and that

(3.10) $$n^{a-1}J(L(n), a) \geq 2^{-a(\alpha+\beta)}(\lambda + z)^{-\beta a}n^{-1}\tau(L(n)).$$

We use the last two inequalities in Section 4.

Finally we establish the convergence in distribution of $\{J(L(n), 1) - f(z, 1)\}\sqrt{n}$ as $n \to \infty$.

<u>Lemma 3.4</u>. Let us assume that Conditions (1.5) and (3.1) hold. Then $\{J(L(n), 1) - f(z, 1)\}\sqrt{n}$ converges in distribution as $n \to \infty$ to

$$m^{-1}\{v - \sigma W(m^{-1}z)\}(1 - z)^{-\alpha}(\lambda + z)^{-\beta} - \beta\Delta I(\lambda > 0)f(z, 1) +$$
$$+ \sigma\int_0^{m^{-1}z} W(p)(1 - mp)^{-\alpha}(\lambda + mp)^{-\beta}\{\alpha(1 - mp)^{-1} - \beta(\lambda + mp)^{-1}\}dp.$$

<u>Proof</u>. Let $R_{L(n),1}(z) = \int_0^{n^{-1}\tau(L(n))}(1 - mp)^{-\alpha}(n^{-1}X_n(0) + mp)^{-\beta}dp - \int_0^{m^{-1}z}(1 - mp)^{-\alpha}(n^{-1}X_n(0) + mp)^{-\beta}dp$, let $R_{n,2}(z) = \int_0^{m^{-1}z}(1-mp)^{-\alpha}(n^{-1}X_n(0) + mp)^{-\beta} - \int_0^{m^{-1}z}(1 - mp)^{-\alpha}(\lambda + mp)^{-\beta}dp$, and **let** $R_{L(n),3}(z) = \int_0^{n^{-1}\tau(L(n))}(1 - mp - V_n(p))^{-\alpha}$

$$(n^{-1}X_n(0) + mp + V_n(p))^{-\beta}dp - \int_0^{n^{-1}\tau(L(n))} (1 - mp)^{-\alpha}(n^{-1}X_n(0) + mp)^{-\beta}dp,$$

$n = 1, 2, \ldots,$ z in $(0, 1)$

We note that

$$(3.11) \qquad J(L(n), 1) - f(z, 1) = R_{L(n),1}(z) + R_{n,2}(z) + R_{L(n),3}(z)$$

Further, let $\theta_n(p)$ be a point in the interval generated by 0 and $V_n(p)$, $n = 1, 2, \ldots,$ p in $[0, 1]$. Finally, let η_n, ν_n be points in the intervals generated by $n^{-1}\tau(L(n))$, $m^{-1}z$ and by 0 and $n^{-1}X_n(0) - \lambda$ respectively, $n = 1, 2, \ldots$.

By the mean value theorem we conclude that

$$(3.12) \qquad R_{L(n),1}(z) = (n^{-1}\tau(L(n)) - m^{-1}z)(1 - m\eta_n)^{-\alpha}(n^{-1}X_n(0) + m\eta_n)^{-\beta},$$

$$(3.13) \qquad R_{n,2}(z) = -\beta(n^{-1}X_n(0) - \lambda)\int_0^{m^{-1}z} (1 - mp)^{-\alpha}(\nu_n + mp)^{-\beta-1}dp, \text{ and that}$$

$$(3.14) \qquad R_{L(n),3}(z) = \int_0^{n^{-1}\tau(L(n))} (1 - mp - \theta_n(p))^{-\alpha}(n^{-1}X_n(0) + mp + \theta_n(p))^{-\beta}$$

$$\{\alpha(1 - mp - \theta_n(p))^{-1} - \beta(n^{-1}X_n(0) + mp + \theta_n(p))^{-1}\}dp.$$

To prove the result of the lemma it suffices to show that

(i) $\sqrt{n}R_{L(n),1}$ converges in distribution to $m^{-1}\{v - \sigma W(m^{-1}z)\}$, that

(ii) $\sqrt{n}R_{n,2}$ converges to $-\beta\Delta I(\lambda > 0)f(z, 1)$ and that

(iii) $\sqrt{n}R_{L(n),3}$ converges in probability to $\sigma\int_0^{m^{-1}z} W(p)(1 - mp)^{-\alpha}(\lambda + mp)^{-\beta}$ $\{\alpha(1 - mp)^{-1} - \beta(\lambda + mp)^{-1}\}dp.$

First, we note that (i) follows by Condition 3.1, by Corollary 3.2 and by Statement (3.12).

Next, we verify (ii). For all p in $(0, m^{-1}z)$ $(1 - mp)^{-\alpha}(\lambda + mp + \nu_n)^{-\beta} \leq (1 - z)^{-\alpha}(\lambda + mp + \nu_n)^{-\beta}$. Thus, by Vitalis theorem [Loève (1963), p. 162] and Condition (1.5) (ii) holds for $\lambda > 0$. Let us assume that $\lambda = 0$. Then

$$0 \leq n^{-1}X_n(0)\int_0^{m^{-1}z}(1 - mp)^{-\alpha}(mp + \nu_n)^{-\beta}dp \leq (1 - z)^{-\alpha}n^{-1}X_n(0)\int_0^{m^{-1}z}(mp + \nu_n)^{-\beta}dp.$$

Thus, (ii) follows for the case $\lambda = 0$ by a simple limit argument.

Finally, we verify (iii). By considering the cases $V_n(p) \geq 0$ and $V_n(p) < 0$ one can show that for all p in $(0, n^{-1}\tau(L(n)))$ and almost all n in $\{1, 2, \ldots\}$

$$1 - mp \geq 1 - mp - \Theta_n(p) \geq \min\{1 - mp, (1 - z)/2\}, \text{ and that}$$

$$4\lambda + 2z + mp \geq n^{-1}X_n(0) + mp + \Theta_n(p) \geq (m + 1)p.$$

Further for almost all n $R_{L(n),3} = \int_0^1 I_n(p)(1 - mp - \Theta_n(p))^{-\alpha}$ $(n^{-1}X_n(0) + mp + \Theta_n(p))^{-\beta}\{\alpha(1 - mp + \Theta_n(p))^{-1} - \beta(n^{-1}X_n(0) + mp + \Theta_n(p))^{-1}\}dp$.

Thus, by Vitalis theorem, by Statement (3.3), and by the convergence with probability 1 of $I_n(p)$ (iii) follows. ||

4. Main Result.

Let $t_o = Af(1,1)$, let $\underline{g(t)}$, t in $[0, t_o)$ be the inverse function of $Af(z,1)$, and let $\underline{h(p)} = (1 - mp)^{-\alpha}(\lambda + mp)^{-\beta}\{\alpha(1 - mp)^{-1} - \beta(\lambda + mp)^{-1}\}$, p in (0, 1). Further, let $\underline{Q(t)}$, t in $(0, t_o)$ be a normal rv such that $EQ(t) = \Delta\{1 - \beta m I(\lambda > 0) f(g(t),1)\}$ and that $\operatorname{Var} Q(t) = m^2\delta^2 f(g(t),2) +$

$$+ \sigma^2 m^{-1} g(t)(1 - g(t))^{-2\alpha} (\lambda + g(t))^{-2\beta} -$$

$$-2\sigma^2 m(1 - g(t))^{-\alpha} (\lambda + g(t))^{-\beta} \int_0^{m^{-1}g(t)} ph(p)dp + 2\sigma^2 m^2 \int_0^{m^{-1}g(t)} \int_0^{p} uh(u)h(p)\, du\, dp.$$

Finally, let $\underline{\phi}$ be the distribution function of a standard normal rv, and let $\underline{B(L(n),x)} = (A^{-1}x - J(L(n),1))\delta^{-1}J^{-1/2}(L(n),2)$, n = 1, 2, ..., x in $(-\infty,\infty)$.

In this section we present our main result. We prove that $(n^{-1}X_n(t) - g(t) - \lambda)\sqrt{n}$ converges in distribution as $n \to \infty$ to the normal rv Q(t) for all t in $(0, t_o)$ provided Condition (1.5) holds. We need the following lemma.

Lemma 4.1. Let us assume that Condition (1.5) holds. Then

$$(4.1) \qquad \lim_{n\to\infty} \sup_{-\infty<x<\infty} |P\{S_{n,L(n)} \le x\} - E\phi\{B(L(n),x)\}| = 0.$$

Proof. Let $0 < \epsilon < \min\{m^{-1}z, m^{-1}(1 - z)\}$, and let $I_{n,2}(\epsilon) = I(|n^{-1}\tau(L(n)) - m^{-1}z| \le \epsilon)$, n = 1, 2, ...

We note that for all x in $(-\infty,\infty)$

$$|E(I(S_{n,L(n)} \le x) - \phi\{B(L(n),x)\})(1 - I_{n,2}(\epsilon))| \le$$
$$\le 2P\{|n^{-1}\tau(L(n)) - m^{-1}z| \ge \epsilon\}.$$

Thus, to prove Statement (4.1) it suffices by Corollary 3.2(b) to show that

$$(4.2) \qquad \lim_{n\to\infty} \sup_{-\infty<x<\infty} |E(I(S_{n,L(n)} \le x) - \phi\{B(L(n),x)\})I_{n,2}(\epsilon)| = 0.$$

We proceed to prove Statement (4.2). Let $\mathcal{B}$ be the σ-field generated by $Z_1, Z_2, \ldots$. Then the conditional rv's $T_{n,1}|\mathcal{B}, \ldots, T_{n,\tau(n)}|\mathcal{B}$ are independent.

Thus, by the Berry-Esseen bound [Loève (1963), p. 288] we obtain that

$$\sup_{-\infty<x<\infty} |P\{S_{n,L(n)} \le x | \mathcal{B}\} - \phi\{B(L(n),x)\}| \le$$
$$\le n^{-1/2} C\delta^{-3} E|U_1 - 1|^3 (n^2 J(L(n),3))(nJ(L(n),2))^{-3/2}, \text{ where}$$

C is a positive constant.

Thus, to prove Statement (4.2) it is enough to show that

$$\text{(4.3)} \qquad \lim_{n\to\infty} n^{-1/2} E(n^2 J(L(n),3))\ (nJ(L(n),2))^{-3/2} I_{n,2}(\epsilon) = 0.$$

Finally, we prove Statement (4.3). By Inequalities (3.9) and (3.10) we conclude that for almost all n in $\{1, 2, \ldots\}$

$$(n^2 J(L(n),3))(nJ(L(n),2))^{-3/2} I_{n,2}(\epsilon) \le$$
$$\le (1-z)^{-3\alpha} 2^{3\alpha+3\beta} (\lambda + z)^{3\beta} (m^{-1}z - \epsilon)^{3/2} \int_0^{m^{-1}z+\epsilon} (\lambda/2+p)^{-3\beta} dp.$$

Consequently Statement (4.3) follows. ||

We note that Lemma 4.1 remains valid if Condition (3.1) is replaced by the weaker condition that $\lim_{n\to\infty} n^{-1}L(n) = z$ in $(0, 1)$.

We are ready now to show that $(n^{-1}X_n(t) - g(t) - \lambda)\sqrt{n}$ converges in distribution as $n \to \infty$ to $Q(t)$ for all t in $(0, t_o)$.

<u>Theorem 4.2</u>. Let us assume that Condition (1.5) holds. Then $\{n^{-1}X_n(t) - g(t) - \lambda\}\sqrt{n}$ converges in distribution as $n \to \infty$ to $Q(t)$ for all t in $(0, t_o)$.

<u>Proof</u>. First, we note that $n^{-1}X_n(t) - g(t) - \lambda =$ $\{n^{-1}(X_n(t) - X_n(0)) - g(t)\} + \{n^{-1}X_n(0) - \lambda\}$. Thus, to prove the result of the theorem it suffices by Condition (1.5) to show that $\{n^{-1}(X_n(t) - X_n(0)) - g(t)\}\sqrt{n}$ converges in distribution as $n \to \infty$ to $Q(t) - \Delta$ for all t in $(0, t_o)$.

We proceed to prove the convergence of $\{n^{-1}(X_n(t) - X_n(0)) - g(t)\}\sqrt{n}$. Let v be in $(-\infty,\infty)$, and let $L(n) = [vn^{1/2} + g(t)n]$. Then $P\{(n^{-1}(X_n(t) - X_n(0)) - g(t))\sqrt{n} > v\} = P\{S_{n,L(n)} \le t\} = P\{S_{n,L(n)} \le Af(g(t),1)\}$. Further, we note that $L(n)$ satisfy Condition (3.1) with $z = g(t)$ in $(0,1)$. Thus to prove the result of the theorem it is enough by Lemma 4.1 to show that

$$\lim_{n\to\infty} E\phi\{B(L(n),Af(g(t),1)\} = P\{Q(t)-\Delta>v\}. \tag{4.4}$$

Finally, we prove Statement (4.4). Let V be a standard normal rv independent of $Z_1, Z_2, \ldots$, and of $U_1, U_2, \ldots$. Then $E\phi\{B(L(n),Af(g(t),1)\} = P\{V\le\delta^{-1}(f(g(t),1)-J(L(n),1))J^{-1/2}(L(n),2)\}$. To prove Statement (4.4) it is enough to show by Lemma 3.3 that

$$\begin{aligned}\lim_{n\to\infty} P\{V\delta f^{1/2}(g(t),2) \le (f(g(t),1) - J(L(n),1))\sqrt{n}\} \\ = P\{Q(t) - \Delta>v\}.\end{aligned} \tag{4.5}$$

By Lemma 3.4 and by the independence of V and $W_n(p)$ for all n in $\{1, 2, \ldots\}$ and for all p in $[0,1]$ we obtain that $V\delta f^{1/2}(g(t),2) + (J(L(n),1)-f(g(t),1))\sqrt{n}$ converges in distribution as $n \to \infty$ to the rv
$V\delta f^{1/2}(g(t),2) - m^{-1}\sigma W(m^{-1}g(t))(1 - g(t))^{-\alpha}(\lambda + g(t))^{-\beta} + \sigma \int_0^{m^{-1}g(t)} W(p)h(p)dp - \beta\Delta I(\lambda>0)f(g(t),1) + m^{-1}v$. Consequently the result of the theorem follows by the linearity property of multivariate normal distributions. ||

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UNCLASSIFIED

Security Classification of this Page

REPORT DOCUMENTATION PAGE

1. REPORT NUMBERS	2. GOVT. ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
FSU No. M495R USARO No. D-41		

4. TITLE On the Normal Convergence of a Class of Simple Batch Epidemics	5. TYPE OF REPORT & PERIOD COVERED Technical Report 6. PERFORMING ORG. REPORT NUMBER FSU Statistics Report M495R
7. AUTHOR(s) Naftali A. Langberg	8. CONTRACT OR GRANT NUMBER(s) USARO DAAG29-79-C-0158
9. PERFORMING ORGANIZATION NAME & ADDRESS The Florida State University Department of Statistics Tallahassee, Florida 32306	10. PROGRAM ELEMENT, PROJECT, TASK AREA AND WORK UNIT NOS.
11. CONTROLLING OFFICE NAME & ADDRESS United States Army Research Office Durham, P.O. Box 12211 Research Triangle Park, N.C. 27709	12. REPORT DATE October 1979 13. NUMBER OF PAGES 14
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)	15. SECURITY CLASS (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release: distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS

Simple batch epidemics, weak convergence, convergence in distribution, normal distributions, Brownian motion, and the Berry-Esseen bound.

20. ABSTRACT

A group of n susceptible individuals exposed to a contagious disease is considered. It is assumed that at each instant in time one or more susceptible individuals can contract the disease.

sub n infinity

The progress of this epidemic is modeled by a stochastic process $X_n(t)$, t in $(0, \infty)$ representing the number of infective individuals at time t. It is shown that $X_n(t)$, with the suitable standartization and under a mild condition, converges in distribution as $n \to \infty$ to a normal random variable for all t in $(0, t_0)$, where t_0 is an identifiable number.

approaches infinity t sub 0